Non-trivial fixed point structure of the two-dimensional ±J 3-state Potts ferromagnet/spin glass

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The fixed point structure of the 2D 3-state random-bond Potts model with a bimodal (±J) distribution of couplings is for the first time fully determined using numerical renormalization group techniques. Apart from the pure and $T = 0$ critical fixed points, *two* other non-trivial fixed points are found. One is the critical fixed point for the random-bond, but unfrustrated, ferromagnet. The other is a bicritical fixed point analogous to the bicritical Nishimori fixed point found in the random-bond frustrated Ising model. Estimates of the associated critical exponents are given for the various fixed points of the random-bond Potts model.

Over the past decades the study of random systems has attracted considerable attention. In particular the possibility of a spin-glass phase has been studied in great detail, as a result of which it now seems well established that no finite temperature equilibrium phase transition to a spin-glass phase occurs in two dimensions [1]. However, even in the absence of a glassy phase a rich fixed point structure can occur. In particular the transition to an ordered phase in the presence of disorder can be controlled by new random fixed points. Recently, it has been suggested that while pure systems can display critical behavior in many different universality classes, the number of universality classes may be more limited when disorder is present [2]. In addition, current progress in the understanding of quantum phase transitions in disordered systems as well as other complex systems has shown that many of these models are in the same universality class as certain disordered classical statistical mechanical models [3]. It is therefore highly desirable to undertake a careful investigation of the complete fixed point structure of some key models, determining all stable and *unstable* fixed points. While considerable progress towards the understanding of random fixed points has been made using analytical techniques [4–7] and series expansions [8], it would seem that a first step towards such a classification of the fixed point structure would have to come from numerical work that does not rely on perturbative methods and where frustration can be taken into account satisfactorily. McMillan [9] investigated the fixed point structure of a two-dimensional, frustrated random-bond Ising model using the domain wall renormalization group (DWRG) method. In this case a bicritical point occurs, separating the ferromagnetic-to-disordered critical phase boundary into two parts. The part at higher temperature and weaker disorder is governed by the pure critical fixed point, while the lower-temperature portion of the phase boundary is governed by the zero-temperature fixed point that also governs the ferromagnetic-to-spin glass transition at zero temperature. For the random

Ising model this bicritical point occurs on the so-called Nishimori line [10], where there is a special gauge symmetry, and the ferromagnetic and spin-glass correlations and susceptibilities are identical [8].

In the present paper we apply a variant of the renormalization group (RG) method of McMillan [9] to the 3-state Potts model and determine all the fixed points. For this model it is known from the Harris criterion [11] that the pure fixed point is unstable against disorder and the simplest possible scenario for the RG flow would be a flow out from the pure fixed point that goes to the zero-temperature fixed point separating the ferromagnetic and spin-glass phases. However, we show that this is *not* the case and that *two* additional non-trivial random fixed points occur at nonzero disorder and nonzero temperature. In agreement with analytical work using only ferromagnetic (unfrustrated) disorder [4,6], an unfrustrated random critical fixed point is found. However, in addition to this critical fixed point a bicritical fixed point *also* occurs in this model, even though the model does *not* have the gauge symmetry that is used to define the Nishimori line in the Ising model. We believe that this fixed point structure is of a rather general nature, in the sense that it should be similar for many other models for which disorder is relevant.

The Hamiltonian we use is:

$$H = - \sum_{<i,j>} J_{ij} h(n_i, n_j), \quad n_i = 1 \ldots q, \quad (1)$$

with $h(n_i, n_j) = \cos(2\pi(n_i - n_j)/q)$, and $q = 2$ (Ising) or $q = 3$ (Potts). The sum is over all nearest-neighbor pairs of sites on a square lattice. We consider the situation where the bonds J_{ij} in Eq. 1 are distributed randomly, and given by a quenched biased bimodal probability distribution: $\mathcal{P}(J) = x\delta(J - 1) + (1 - x)\delta(J + 1)$. As an application of this type of model, it has been suggested that the orientational freezing in molecular glasses, such as N_2-Ar and KBr-KCN, can be partially described by a three-dimensional 3-state Potts spin glass [12].

The way we apply the DWRG is slightly novel and we therefore begin by reviewing the method and the scaling ideas behind our approach. The DWRG as proposed by McMillan [9] estimates the singular part of free energy by calculating the domain wall energy, $[\Delta F]$, and its standard deviation, $\sigma(\Delta F)$, for systems of size $L \times L$. Here $[\cdot]$ denotes disorder averaging. We calculate the total free energy difference between periodic (p) and twisted (t) boundary conditions, $\delta F_m = F_{\rm t} - F_{\rm p}$, for a long cylinder of size $L \times M$ where $M = Lm$, and m is a large integer running up to of order $10^5 - 10^6$. The proper generalization of twisted boundary conditions to $q > 2$ is to change the local Hamiltonian across the boundary in the L-direction as follows [13]: $h(n_1, n_L) = \cos(2\pi((n_1 - n_L + 1)/q)$. This change is done at a seam going *along* the entire length of the cylinder. ΔF for each $L \times L$ subsystem is simply defined as $\Delta F = \delta F_m - \delta F_{m-1}$. The free energies are evaluated exactly using standard transfer matrix techniques [14] and the only errors in the calculation therefore stem from the incomplete disorder averaging.

Pure systems. In the absence of disorder it is known from hyperscaling that the singular part of the free energy density scales as the inverse of a correlation volume

$$\frac{f_s}{k_b T_c} = \frac{C}{\xi^d}. \qquad (2)$$

Since $f_s \sim \Delta F/L^d$ it follows that an appropriate finite-size scaling form for ΔF is

$$\frac{\Delta F}{k_b T} = Ag(\delta L^{1/\nu}), \qquad (3)$$

with g a universal function and $\delta = |T - T_c|$. Hence, the critical point, T_c, can be located by standard methods, i.e. by tracing ΔF as a function of T for several different system sizes and locating the point where the lines cross. Here A is the *universal* amplitude for the spin stiffness of the system. The universality of A has been investigated extensively at finite temperature transitions [13,15–18], and is known exactly for the *pure* $q = 2, 3, 4$ Potts models as a function of the aspect ratio $s = 1/m$ [13]. In the limit $s \to 0$ it follows from conformal invariance that $A = \pi\eta$ [16], where η is the magnetic critical exponent. Since we use an aspect ratio of $s = 1/m$, essentially zero, we see that $\frac{\Delta F}{k_b T_c} = \pi\eta$. Hence, ΔF directly measures a *bulk* critical exponent.

Disordered systems. Just as $\Delta F/k_B T$ is a *universal* amplitude at the pure fixed point it is natural to assume that any random fixed point occurring at *finite, nonzero* temperature and disorder will be characterized by a universal *distribution* of $\Delta F/k_B T$. In particular, the mean $[\Delta F]/k_B T$ and the standard deviation $\sigma(\Delta F)/k_B T$ are then universal [19,20]. The generalization of the finite size scaling relation Eq. (3) to include disorder is therefore:

$$\frac{[\Delta F]}{k_b T} = g_X(\delta L^{1/\nu}, \epsilon L^{\lambda_2}), \qquad (4)$$

valid close to any finite temperature fixed point X at (x^*, T^*), with g_X a universal scaling function that is, however, different for each distinct fixed point X. $\sigma(\Delta F)/k_B T$ obeys a similar finite-size scaling form. Here δ is a linear combination of $\Delta x = |x - x^*|, \Delta T = |T - T^*|$, that is the eigenvector of the RG flow that moves one away from the phase boundary; $1/\nu$ is the corresponding leading eigenvalue. ϵ is another linear combination that is the next subrelevant ($\lambda_2 > 0$) or irrelevant ($\lambda_2 < 0$) eigendirection of the RG flow that is tangent to the phase boundary. Thus we can estimate the fixed points $(x^\star, T^\star)$ as the points in the phase diagram where $([\Delta F], \sigma(\Delta F))$ are independent of L. We obtain a picture of the RG flow by measuring $[\Delta F]$ and $\sigma(\Delta F)$ at various (x, T) points and seeing how they vary as L is increased at constant x and T.

Zero temperature. If the transition is controlled by a zero temperature fixed point Z, then hyperscaling, Eq. (2), is *not* valid, and we cannot use Eq. (3) as our starting point. The appropriate finite-size scaling form is instead

$$[\Delta F] = L^\theta g_Z(\delta L^{1/\nu}, \epsilon L^{-\theta}), \qquad (5)$$

with $\theta > 0$. Hence, $[\Delta F]$ is *not* universal along the part of the phase boundary where the flow is controlled by a zero-temperature fixed point. Instead, $[\Delta F]$ grows with increasing L as L^θ at the critical point. At small temperatures we expect that $\delta \sim |x - x_c|$ and $T \sim \epsilon$, i.e. that the disorder x is the relevant flow direction. A positive θ implies that the flow is *into* the zero-temperature fixed point and hence that temperature is an irrelevant variable.

We now proceed as follows: In order to accommodate both finite temperature and zero temperature fixed points in the same plot we consider the two quantities $r = \sigma(\Delta F)/[\Delta F]$ and $f = k_b T/[\Delta F]$. Any fixed point should be characterized by universal values of these quantities $(r^\star, f^\star)$. In particular we know that for the pure fixed point we have $(r^\star, f^\star) = (0, (\pi\eta)^{-1})$. A ferromagnetic phase will be characterized by $(r, f) \to (0, 0)$ whereas in the paramagnetic phase $[\Delta F] \to 0$. we can now directly determine the complete fixed point structure by calculating $(r(L), f(L))$ for different values of L on a grid of (x, T) close to the phase boundary. By connecting points, $(r(L), f(L))$ corresponding to L and L' calculated at the same (x, T), with an arrow terminating at the larger of L, L', the flow becomes clearly visible and "fixed points" $(r^\star, f^\star)$ appear where *both* f and r are independent of L. The linear flow around a fixed point should be characterized by

$$\begin{pmatrix} f(L') - f^\star \\ r(L') - r^\star \end{pmatrix} = M \begin{pmatrix} f(L) - f^\star \\ r(L) - r^\star \end{pmatrix}. \qquad (6)$$

Through a least-square fit to this equation for a number of points close to the fixed point the matrix M as well as $(r^\star, f^\star)$ can be determined and consequently the eigenvalues, $\lambda_1 = 1/\nu$, λ_2, and eigenvectors of the linear flow can be determined. The magnetic exponent η is estimated by exact calculations of $[<m^2>] \sim L^{2-\eta}$, the square of the magnetization per site, directly at the fixed point.
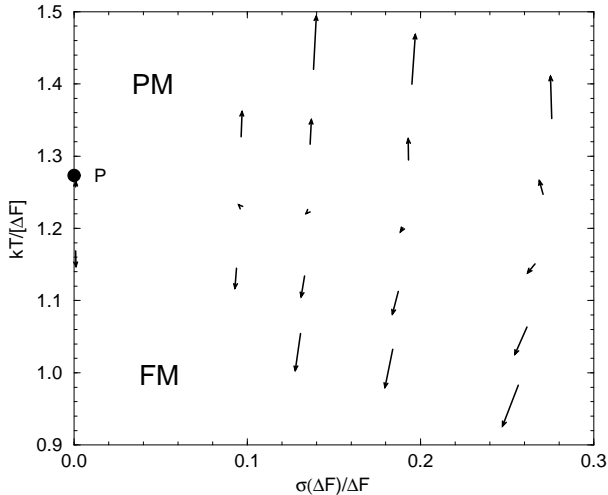

figure 1: The flow close to the pure fixed point for the 2D Ising model with bimodal ferromagnetic disorder $\mathcal{P}(J_{ij}) = (1-x)\delta(J_{ij}-1) + x\delta(J_{ij}-0.3)$. P is the exactly known pure fixed point. In most cases 1 million $L \times L$ blocks were measured.

In order to establish that our approach is sensitive even to *marginal* flow we first consider the two-dimensional Ising model with *ferromagnetic* disorder. In this case it seems well established [6,4,5] that this type of disorder is *marginally irrelevant* at the pure critical point. Hence the flow should be towards the pure critical fixed point P. Our results for this case are shown in Fig. 1 for a bimodal distribution of couplings of strength J and $0.3J$. The arrows connect results for $L = 6$ and $L' = 8$. It is clear that the flow in this case indeed is *towards P*, located at $(0, 4/\pi)$, in agreement with the theoretical studies [6,4,5].

We now turn to a discussion of our main results for the 3-state Potts model. The flow diagram is shown in Fig. 2 with $L = 6$ and $L' = 8$. Rather clearly, four different fixed points are visible along the phase boundary of the ferromagnetic phase. They are: the pure critical fixed point P, the random critical fixed point S, the bicritical fixed point N and the zero-temperature fixed point Z. Close to the latter fixed point our calculations have been performed exactly at zero temperature. The RG flow along the phase boundary is away from P and N and towards S and Z. Also indicated (bold arrows) in Fig. 2 are the numerically determined RG eigenvectors at
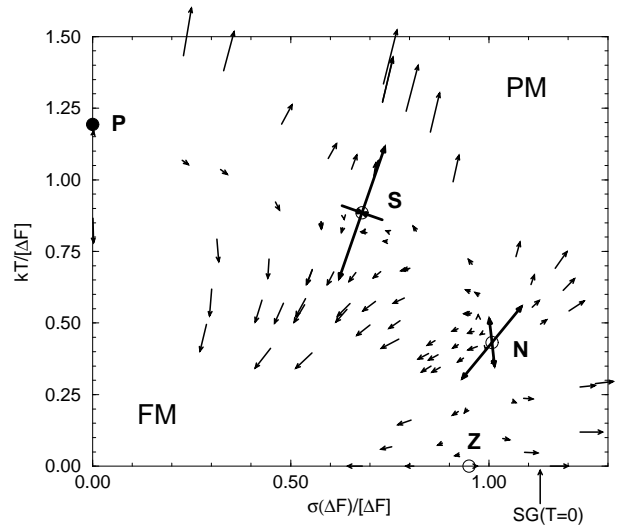

figure 2: The flow diagram for the q=3 Potts model. P is the exactly known pure critical fixed point, S the random critical fixed point, N the analogue of the Nishimori bicritical fixed point, and S is the zero temperature fixed point. The bold arrows indicate the numerically determined RG eigenvectors. The **PM** and **FM** indicate the paramagnetic and ferromagnetic phase. In most cases 200,000 $L \times L$ blocks were measured.

the fixed points S and N. At the *stable fixed point* (S), with approximately $(x^\star, T^\star) \sim (0.93, 1.16)$ $(r^\star, f^\star) = (0.68(9), 0.89(3))$ we find $\lambda_1 = 0.94 - 1.04$, $\lambda_2 \sim -.2$, and hence $\nu = 1/\lambda_1 \sim 0.96 - 1.06$. Using $[<m^2>]$ we find $\eta = 0.23 - 0.28$. Both of these exponents are in good agreement with analytical results [4,6] as well as Monte Carlo work [21], for the $q = 3$ Potts model with ferromagnetic disorder, so S appears to be the same random fixed point as in the non-frustrated random ferromagnet q=3 Potts model. The *bicritical fixed point* (N) at $(x^\star, T^\star) \sim (0.88, 0.68)$, $(r^\star, f^\star) = (1.01(3), 0.43(3))$ is characterized by the eigenvalues $\lambda_1 = 0.55 - 0.65$, $\lambda_2 = 0.30 - 0.40$ using the lattice sizes $L = 6$ and 8. However, at this fixed point a standard scaling plot of f approximately along the eigendirection of λ_1 and including $L = 10$ yields $1/\nu \sim 0.75 - 0.8$. Hence, we estimate $\nu = 1.28 - 1.36$, and using $[<m^2>]$ we find $\eta = 0.17 - 0.22$. Both of these exponents are remarkable similar to the ones determined at the bicritical point in the 2D random Ising model [8] of $\nu = 1.32(8), \eta = 0.20$. The *zero temperature fixed point* (Z) located at $x^\star \sim 0.88(1)$, $r^\star \sim 0.96$ and we determine the exponents $\nu = 1.45 - 1.55, \eta = 0.18(1)$, where we for the Ising model find $x^\star = 0.89(1)$, $\nu = 1.35 - 1.45, \eta = 0.18(1)$, in agreement with McMillan [9]. At this fixed point we also estimate $\theta \sim 0.3$. Since this exponent is *positive*, temperature is irrelevant at this fixed point.

Several conclusions can now be drawn for the first

time for the randomly frustrated 3-state Potts model from the results in Fig. 2. The random critical fixed point S has exponents that to within our precision appear close to those of the pure Ising model, and in good agreement with theoretical predictions [4,6]. However, we find $\sigma(\Delta F) \neq 0$ at S, which certainly distinguishes it from the pure Ising critical fixed point. At S we have also determined the fraction of negative ΔF for the fixed point distribution. This fraction, which is a measure of the "renormalized" frustration level, is decreasing with increasing L at S indicating that the fixed point is the same as for unfrustrated ferromagnetic disorder. A new bicritical fixed point N appears even though the model is not gauge-symmetric [10]. This fixed point has exponents similar to the bicritical fixed point for the Ising model [8]. At the zero-temperature fixed point Z temperature is irrelevant as it is for the Ising model. The critical concentration x_c is at this point identical for the Ising and $q = 3$ Potts model to within the numerical resolution. Recently, Nishimori [22] has rigorously established that $x_c^{\text{non}-\text{Ising}} \geq x_c^{\text{Ising}}$ for any bimodal vector-spin model that is invariant under the operation $\mathbf{S_i} \rightarrow -\mathbf{S_i}$. However, the $q = 3$ Potts model is *not* invariant under this operation and it is therefore remarkable that we for the $q = 3$ model find that the inequality apparently is satisfied as an equality. Finally, it would appear that the $q = 3$ Potts model is slightly reentrant in Fig. 2, as occurs for the same model on hierarchical lattices [23]. However, here, we believe that this is due to the fact that corrections to scaling are more pronounced at low temperatures, in particular for a bimodal distribution. It seems also possible that the phase diagram could show reentrance in (r, f) space but not in (x, T) space. For the bimodal Ising model it is known that the phase boundary is vertical [24] down from the bicritical point (N) where the Nishimori line [10] intercepts the phase boundary. Remarkably, this is also what we find in the (x, T) space for the $q = 3$ Potts model. Summarizing, our results indicate that at N and Z the $q = 2$ (Ising) and $q = 3$ *random* bimodal Potts models are remarkably similar. However, for ferromagnetic disorder the Ising model does not have a stable random fixed point at finite temperature [4,6]. The stable random fixed point (S) of the $q = 3$ model has critical exponents very close to the pure $q = 2$ Ising model. In light of these observations we believe that the fixed point structure shown in Fig. 2 is rather general for two-dimensional disordered models when disorder is relevant.

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